Geophysical Fluid Dynamics: from the Lab, up and down!

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UNIVERSITÉ Grenoble Alpes

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Lecture 1
Overview and basic concepts

FDEPS

Kyoto, November 27, 2018

Overview

1. Basic concepts
2. Mantle convection and plate tectonics
3. Core dynamics and the geodynamo
4. Turbulence in planetary cores
5. The formation of planets

Laboratory experiments

1. Basic concepts

Outline

1. Basic concepts

    1.1. Equations

    1.2. Dimensional analysis

    1.3. Convection and pattern formation

    1.4. Earth's interior

1.1. Equations

Building blocks

- Newton's laws of mechanics
- Maxwell's equations of electrodynamics

Fondamental interactions

- Continuum mechanics : internal stress, electric current, etc
- Thermodynamic principles : Fourier's law, second law, etc

Statistical physics

- Equations of state : $\rho(P, T, \chi)$
- Transport properties : ν, κ, σ, etc

ab initio Schrödinger equation + material science experiments

Our set of equations

- Navier-Stokes or momentum equation
- Heat or entropy equation
- Magnetic induction equation (from Maxwell equations)

Navier-Stokes equation

- mass conservation equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

- Navier-Stokes or momentum equation

$$\rho \frac{D\mathbf{u}}{Dt} = \mathbf{f} \quad \text{with} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad \text{the material derivative}$$

Density ρ, fluid velocity $\mathbf{u}$, forces $\mathbf{f}$.

The Boussinesq approximation

- **All material properties are uniform and constant - except density in the buoyancy force**, in which its temperature variation is retained:

$$\rho(T) = \rho_0 \left(1 - \alpha(T - T_0)\right)$$

where α is the coefficient of thermal expansion

Boussinesq Navier-Stokes equation

- continuity equation for an incompressible fluid:

$$\nabla \cdot \mathbf{u} = 0$$

- Navier-Stokes equation for a fluid in a rotating frame within a gravity field and a magnetic field

$$\underbrace{\rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right)}_{inertia} + \underbrace{2\rho_0 \boldsymbol{\Omega} \times \mathbf{u}}_{Coriolis} = \underbrace{-\nabla P}_{pressure} + \underbrace{\rho_0 \big(1 - \alpha(T - T_0)\big)\, \mathbf{g}}_{buoyancy} + \underbrace{\mathbf{j} \times \mathbf{B}}_{Lorentz} + \underbrace{\mu \nabla^2 \mathbf{u}}_{viscous}$$

Frame rotation vector $\boldsymbol{\Omega}$, pressure P, gravity $\mathbf{g}$, electric current density $\mathbf{j}$, magnetic field $\mathbf{B}$, dynamic viscosity μ.

- Heat equation in the Boussinesq approximation

$$\underbrace{\rho_0 C_P \left(\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T \right)}_{advection} = \underbrace{H}_{production} + \underbrace{k \nabla^2 T}_{diffusion}$$

Heat capacity C_P, heat production per unit mass H, thermal conductivity k.

Maxwell equations

- Gauss law for magnetism $$\nabla \cdot \mathbf{B} = 0$$
- Faraday's law of induction $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
- Ampère's law $$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$
- Ohm's law $$\frac{\mathbf{j}}{\sigma} = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$

Electric field **E**, magnetic permeability μ_0, electric permittivity ε_0, electric conductivity σ.

Magnetic induction equation

- One can neglect the displacement current $\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$ when the

fluid velocity u is very small compared to the speed of light c: this is the magnetohydrodynamics (MHD) approximation.

- Combining the four preceding equations, one gets the **magnetic induction equation** in the MHD approximation (assuming uniform electric conductivity)

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{induction} + \underbrace{\eta \nabla^2 \mathbf{B}}_{diffusion}$$

Magnetic diffusivity $\eta = 1/\mu_0\sigma$.

1.2. Dimensional analysis

Dimensional analysis: what for?

- figure out the **dominant terms** in our equations
- reduce the number of relevant parameters —> **dimensionless numbers**
- **permit reduced scale laboratory experiments**
- use identical mathematical tools for different physical problems

A simple example

- Consider the magnetic induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

- Let the flow be at a typical length-scale L, with a typical velocity U (from which we derive a typical time $T=L/U$). Let's take B as a typical magnetic scale (though we don't even need it).

- Rewriting our equation in terms of dimensionless variables $\mathbf{u}^*=\mathbf{u}/U$, $t^*=t/T$, etc, we get:

$$\frac{\partial \mathbf{B}^*}{\partial t^*} = \nabla^* \times (\mathbf{u}^* \times \mathbf{B}^*) + \frac{\eta}{UL} \nabla^{*2} \mathbf{B}^*$$

A simple example

- One dimensionless number appears in our dimensionless equation: the **magnetic Reynolds number**:

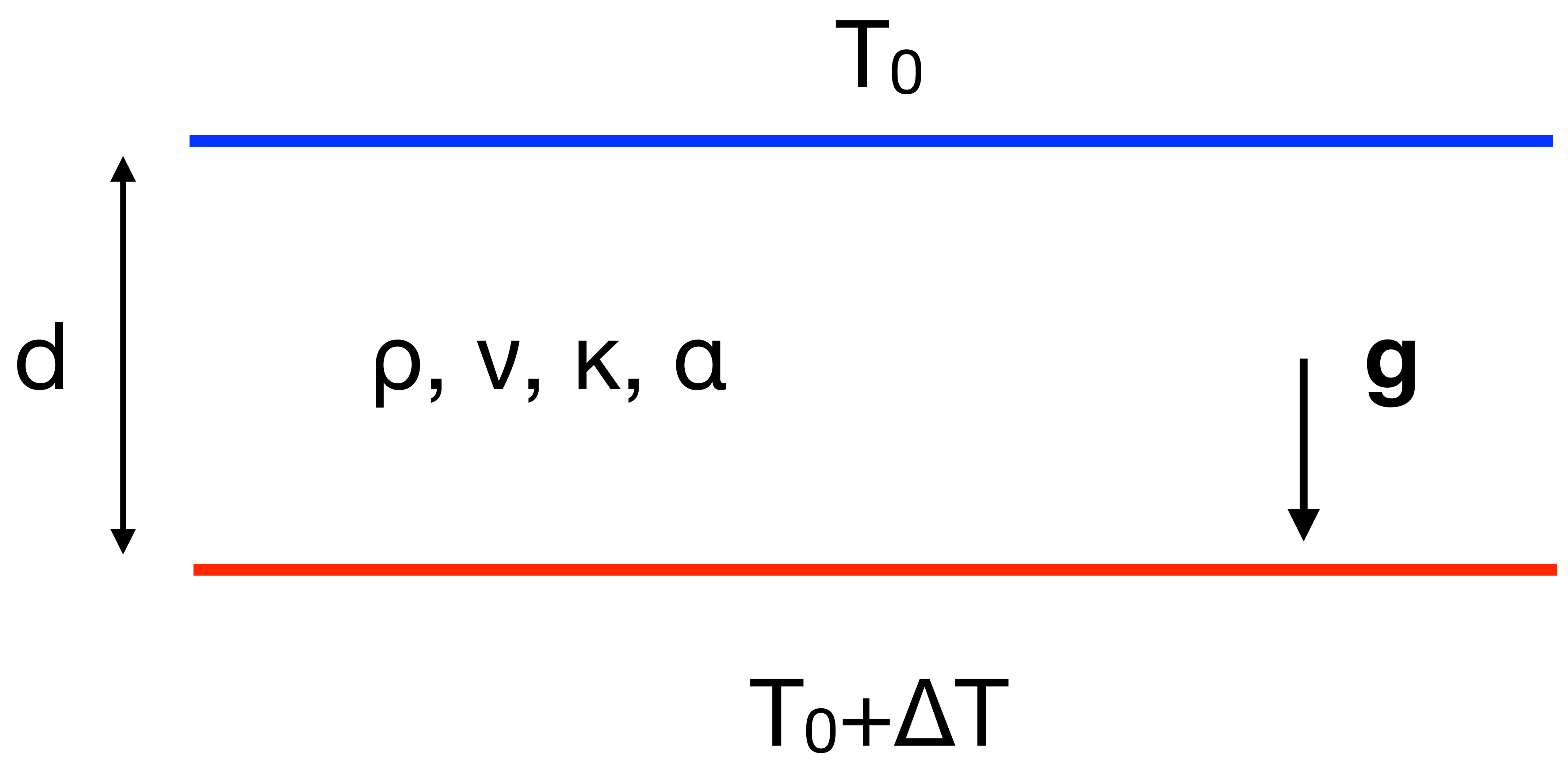
$$Rm = \frac{UL}{\eta}$$

- We have reduced the number of relevant parameters.

- To observe the same physics, it suffices to keep this dimensionless number identical between two systems with otherwise differing U, L and η.

A classical example: Rayleigh-Bénard convection

- The Rayleigh-Bénard (R-B) problem: what controls the formation of convection cells in a fluid sandwiched between two horizontal plates, the lower one being at a higher temperature than the upper one?



A classical example: Rayleigh-Bénard convection

- In the Boussinesq approximation, the equations governing this problem can be written as:

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \\ \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\dfrac{1}{\rho}\nabla P + \big(1 - \alpha(T - T_0)\big)\,\mathbf{g} + \nu \nabla^2 \mathbf{u} \\ \dfrac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T \end{cases}$$

with $\nu = \mu/\rho$ the kinematic viscosity and $\kappa = k/\rho C_P$ the thermal diffusivity.

- Introducing the hydrostatic pressure P_H, whose gradient is $\rho\mathbf{g}$, and *choosing* the following scales:

$$\mathbf{u} = \frac{\kappa}{d}\mathbf{u}^*, t = \frac{d^2}{\kappa}t^*, T = T_0 + \Delta T\, T^*, P = P_H + \frac{\rho\kappa\nu}{d^2}P^*$$

- and multiplying the momentum equation by $d^3/\kappa\nu$, we get the following dimensionless equations:

$$\begin{cases} \dfrac{\kappa}{\nu}\left(\dfrac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^*\right) = -\nabla^* P^* - \dfrac{\alpha \Delta T g d^3}{\kappa\nu} T^* \hat{\mathbf{z}} + \nabla^{*2}\mathbf{u}^* \\ \dfrac{\partial T^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)T^* = \nabla^{*2} T^* \end{cases}$$

where $\hat{\mathbf{z}}$ is the unit vector pointing downwards in the vertical direction.

A classical example: Rayleigh-Bénard convection

which yields **two** dimensionless numbers this time:

The Rayleigh number: $$Ra = \frac{\alpha \Delta T g d^3}{\kappa \nu}$$

The Prandtl number: $$Pr = \frac{\nu}{\kappa}$$

- In principle, in order to obtain similarity between a system such as the Earth's mantle, say, and a lab experiment, we need to keep these two numbers identical.

- However, in the Earth's mantle, the Prandtl number is enormous ($Pr \simeq 10^{23}$). This indicates that the inertial term can be neglected altogether. Therefore, one only needs to keep the Rayleigh numbers identical, as long as the Prandtl number is large enough as well in the experiment.

Buckingham's Π theorem

- What we have just seen works well when we know the governing equations, and if we make the right guesses for the typical scales.

- A more general approach is brought by the 'Pi' theorem of Buckingham (1914). Even when governing equations are unknown, there are some links between the relevant parameters of a physical experiment. In particular any correct physical equation must be **dimensionally homogeneous**.

Buckingham's Π theorem

« *Conclusion*. — A convenient summary of the general consequences of the principle of dimensional homogeneity consists in the statement that any equation which describes completely a relation subsisting among a number of physical quantities of an equal or smaller number of different kinds, is reducible to the form

$$\psi\left(\Pi_1, \Pi_2, \ldots, etc.\right) = 0$$

in which the Π's are all the independent dimensionless products of the form Q_1^x, Q_2^y, etc. that can be made by using the symbols of all the quantities Q.

While this theorem appears rather noncommittal, it is in fact a powerful tool and comparable, in this regard, to the methods of thermodynamics or Lagrange's method of generalized coordinates. It is hoped that the few sample illustrations of its use which have been given will prove interesting to physicists who have not been in the habit of making much use of dimensional reasoning; but if this paper merely helps a little toward dispelling the metaphysical fog that seems to be engulfing us, it will have attained its object. »

Buckingham, 1914

Buckingham's recipe

1. ***Guess*** and decide what might be the n relevant physical quantities.

2. Count the number k of **fundamental *dimensions*** (e.g., *length*, *time*, etc) that come into play.

3. Then, $p = n\text{-}k$ is the number of independent dimensionless products Π_i one can form.

4. One can **calculate** these p dimensionless products by choosing k relevant 'fundamental' physical quantities (out of n), covering the k fundamental dimensions, and expressing the remaining $n\text{-}k$ relevant physical quantities as products of polynomials of the fundamental physical quantities.

Buckingham's recipe applied to R-B convection

1. Choose $n = 7$ relevant physical quantities:

physical quantity	ΔT	ν	κ	α	g	d	C_P
unit	K	m^2s^{-1}	m^2s^{-1}	K^{-1}	ms^{-2}	m	$m^2s^{-2}K^{-1}$

2. There are $k = 3$ fundamental dimensions (*length, time, temperature*).

3. Let's pick d, α, and g as our $k = 3$ fundamental physical quantities.

4. We obtain $p = 4$ dimensionless products Π_i by expressing the 4 remaining physical quantities ΔT, ν, κ, and C_P as a function of these 3.

Buckingham's recipe applied to R-B convection

We thus obtain:

$$\Pi_1 = \alpha \Delta T \qquad \Pi_2 = \frac{\nu^2}{g d^3} \qquad \Pi_3 = \frac{\kappa^2}{g d^3} \qquad \Pi_4 = \frac{C_P}{\alpha g d}$$

which we can easily recombine into 4 more classical dimensionless numbers:

$$Ra = \frac{\alpha \Delta T g d^3}{\kappa \nu} \qquad Pr = \frac{\nu}{\kappa} \qquad Di = \frac{\alpha g d}{C_P} \qquad \epsilon = \alpha \Delta T$$

The last two numbers did not appear in our earlier derivation because we used the Boussinesq approximation. The dissipation number *Di* measures the effect of compression on density and controls viscous dissipation.

1.3. Convection and pattern formation

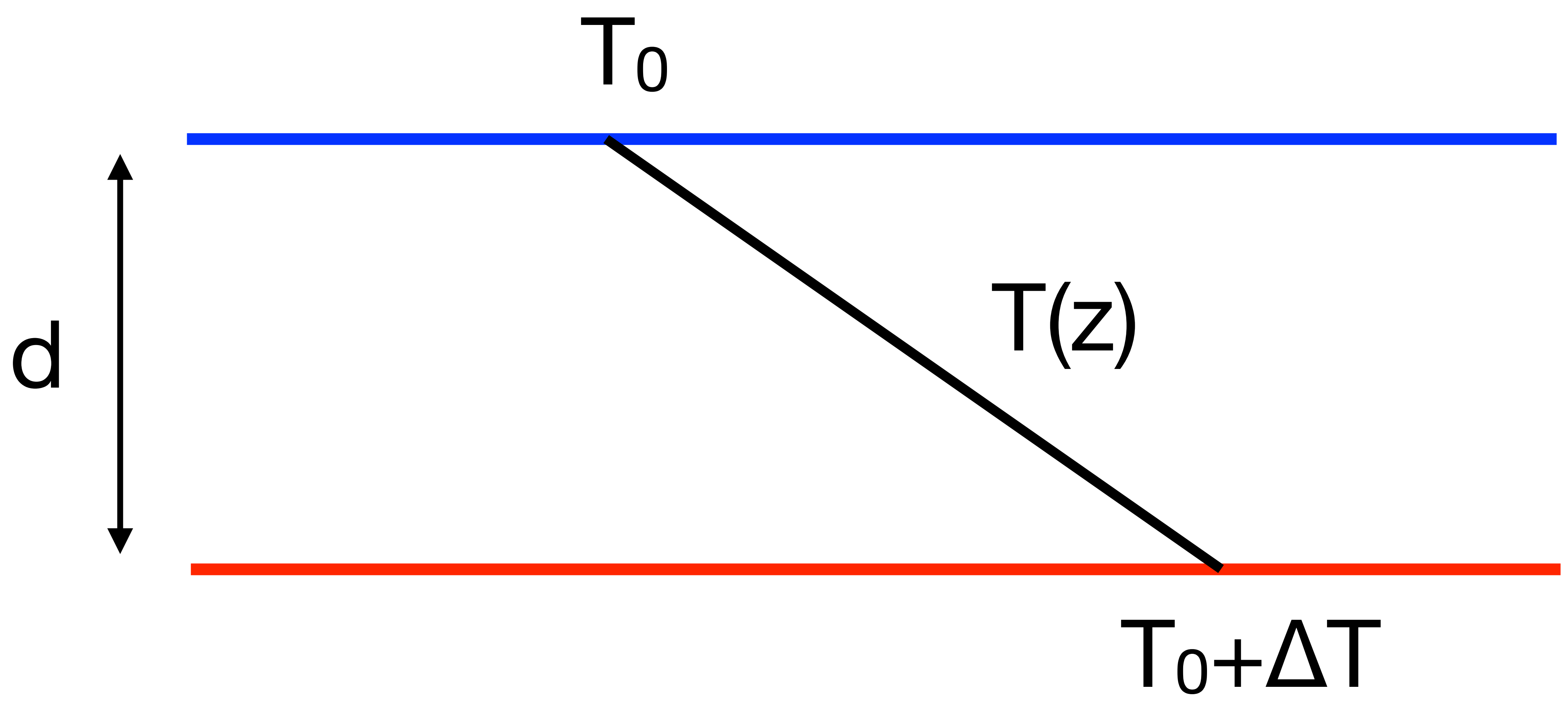
- Let's write again our dimensionless equations for Rayleigh-Bénard convection:

$$\begin{cases} Pr^{-1}\left(\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla P - Ra\,T\,\hat{\mathbf{z}} + \nabla^2\mathbf{u} \\ \dfrac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T = \nabla^2 T \end{cases}$$

- Let's simplify them further, assuming $Pr \gg 1$:

$$\begin{cases} -\nabla P - Ra\,T\,\hat{\mathbf{z}} + \nabla^2\mathbf{u} = \mathbf{0} \\ \dfrac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T = \nabla^2 T \end{cases}$$

- An obvious solution of these equations is $\mathbf{u} = \mathbf{0}$ and $T = T_0 + \frac{z}{d}\Delta T$



and this would be *the* solution if our equations were linear, which is not quite the case because of the $(\mathbf{u} \cdot \nabla)T$ term.

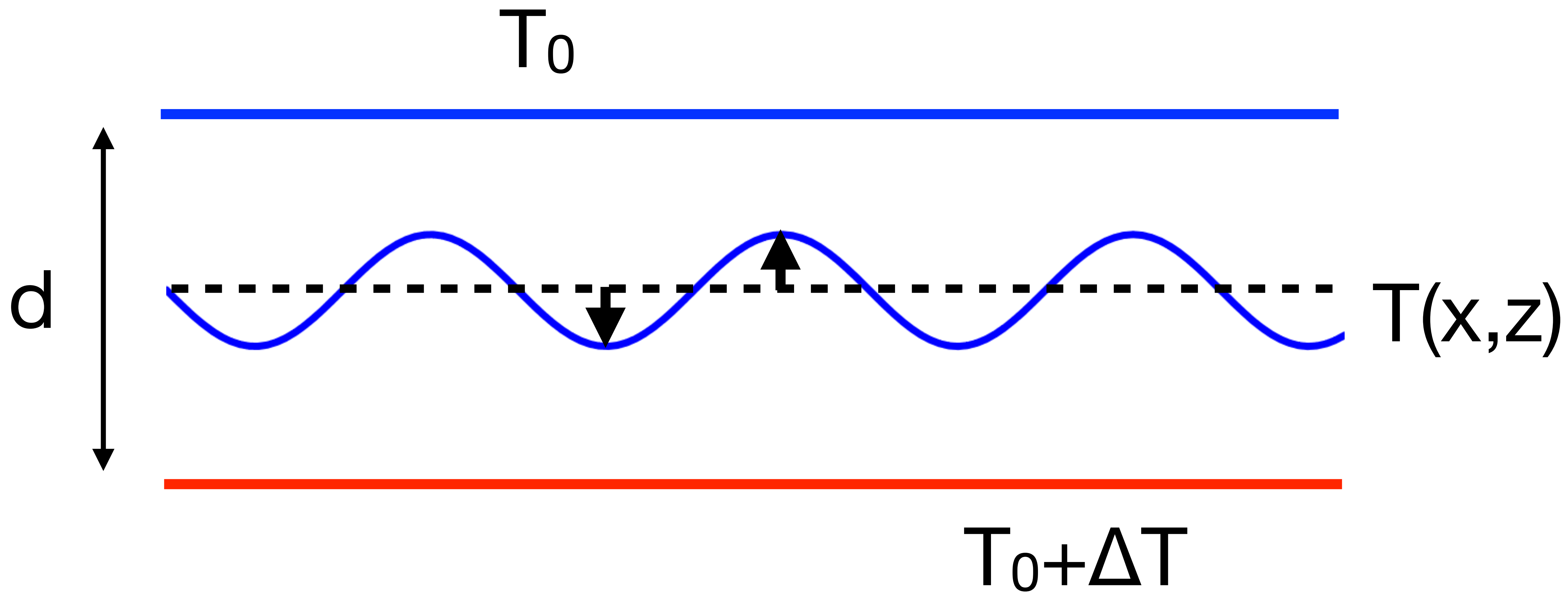
- Non-linear equations can admit an infinite number of solutions.

linear stability analysis

- If we start from this stationary conductive solution and add a small velocity perturbation **u**, it will advect the isotherms.
- The linear stability analysis, pioneered by Lord Rayleigh (1916), allows to test whether such a perturbation will grow or decay *(see the excellent presentation by Prof Chris Jones in last's year FDEPS)*.
- It is found that at least *some* perturbations will grow once the Rayleigh number Ra is higher than some critical value Ra_c.
- $Ra_c = 1708$ for rigid isothermal boundaries.

the inevitability of pattern formation

- In order to conserve mass, the velocity perturbation has to advect the isotherms **up** in some places, and **down** in other places, inevitably leading to **pattern formation**.



- Let's look at some convection patterns from the Lab and elsewhere.

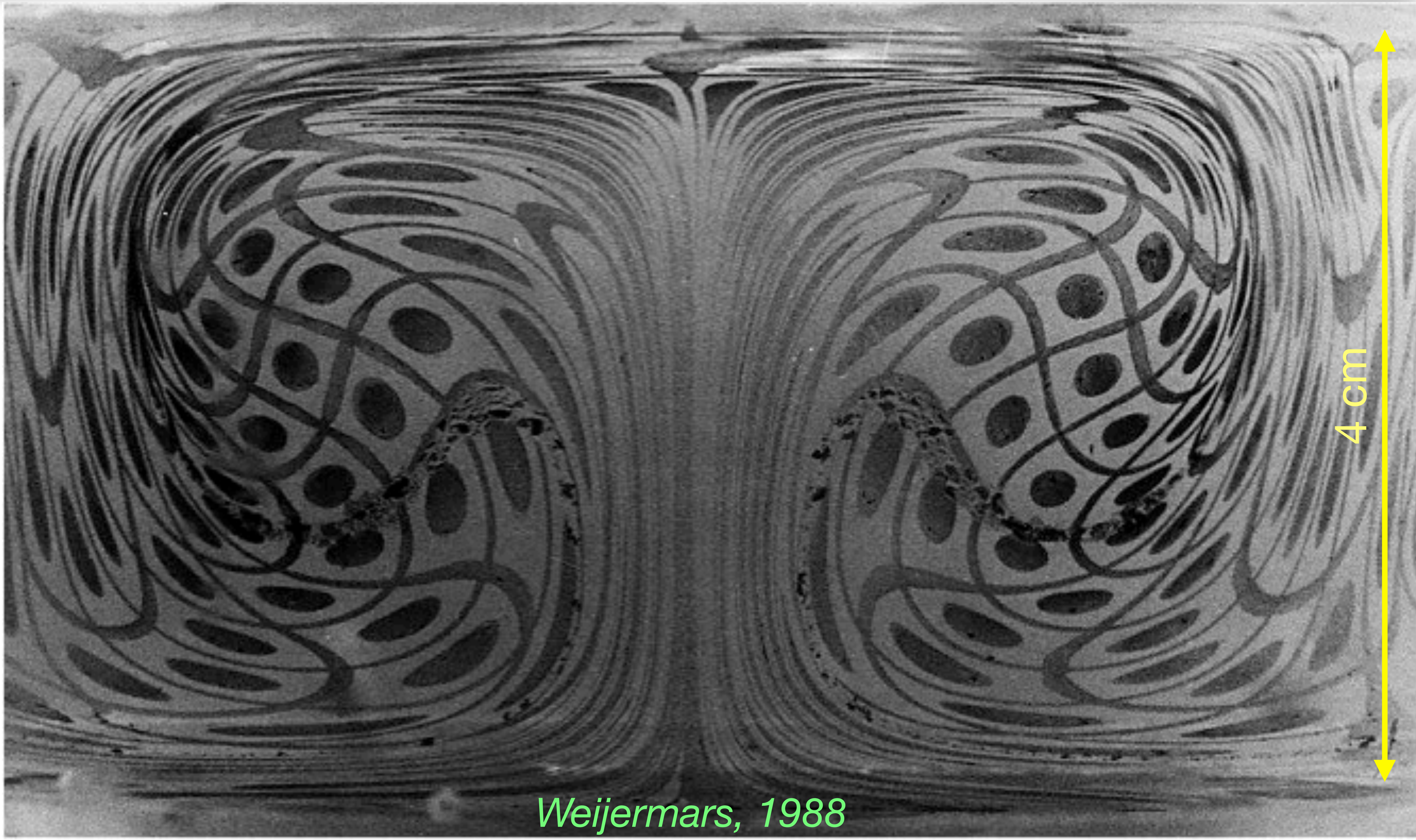
Convection in a centrifuge

$$Pr = \frac{\nu}{\kappa} = 10^6$$

$$Ra = \frac{\alpha \Delta T \gamma d^3}{\kappa \nu}$$

with

$$\gamma \simeq 1000\, g$$



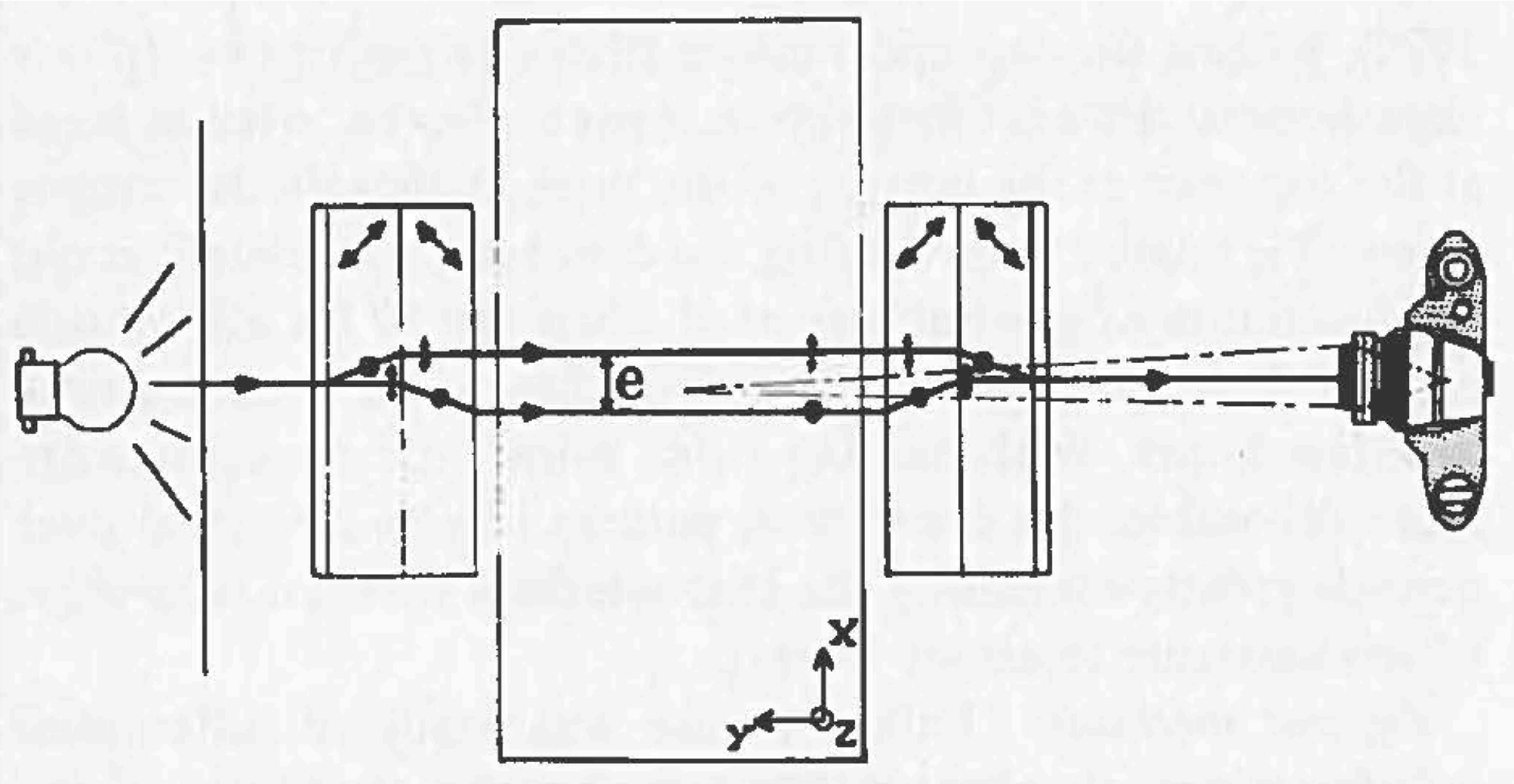
Rayleigh-Bénard convection cells

Horizontal temperature gradient isocontours

Vertical temperature gradient isocontours

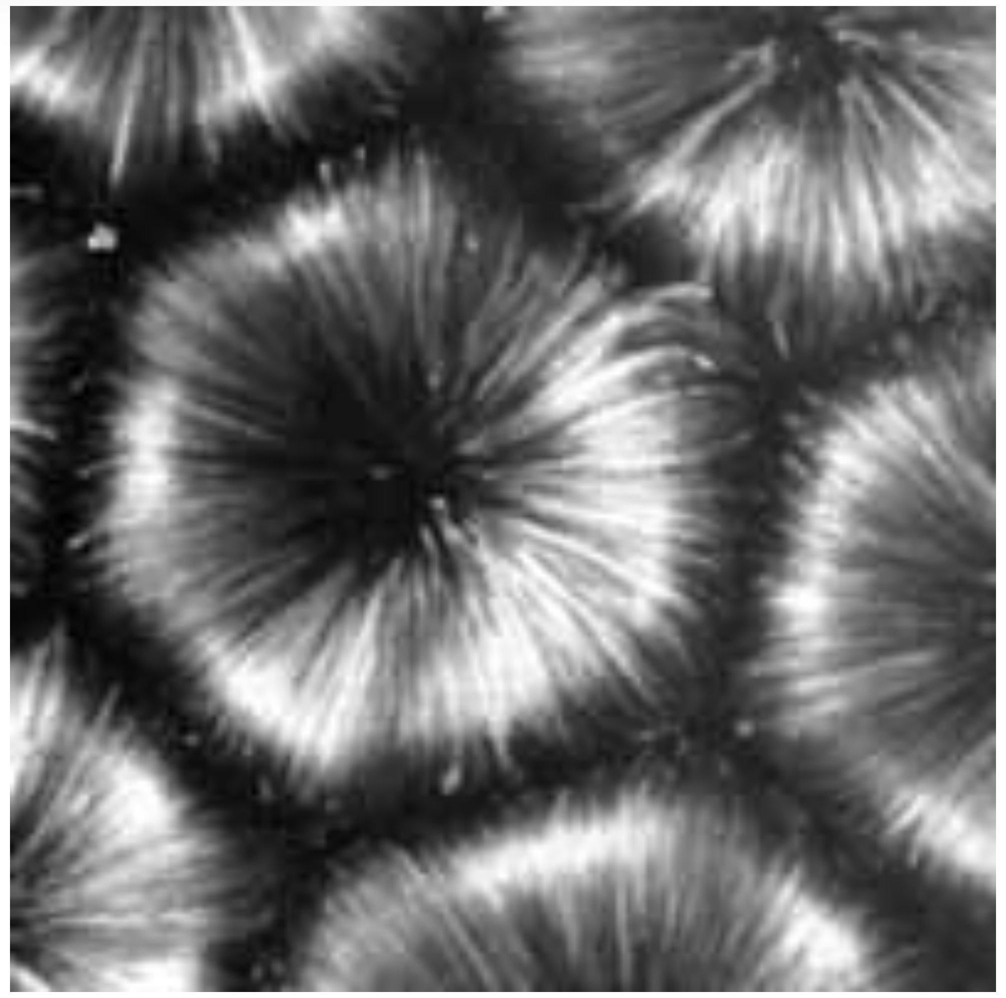
Nataf & Richter, 1982

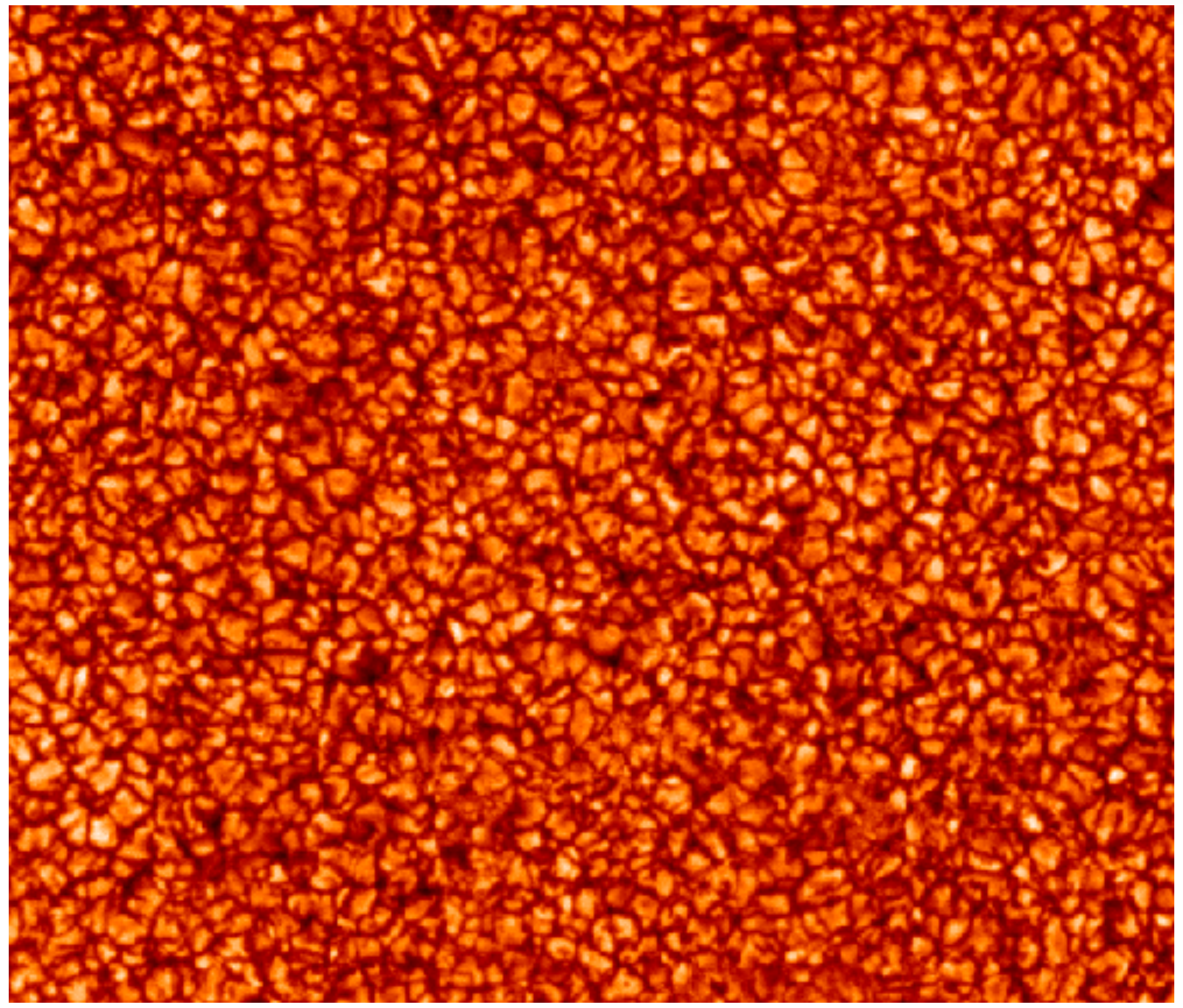
- An experimental visualization technique: differential interferometry

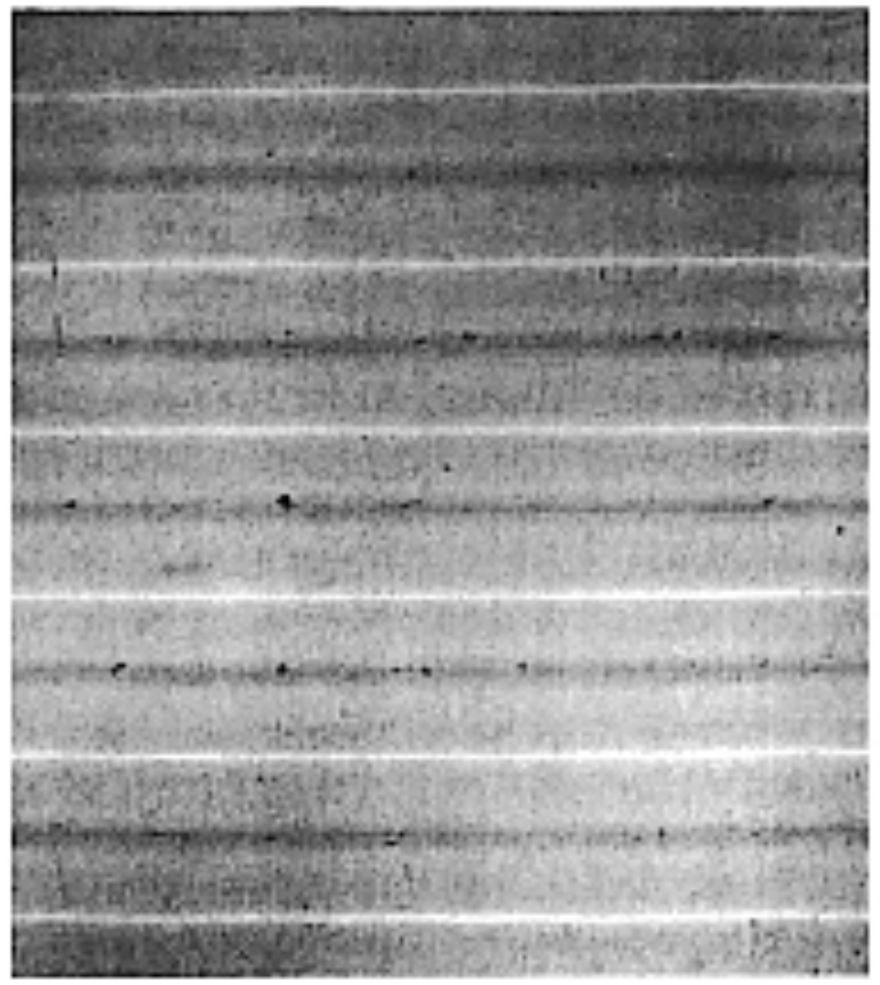


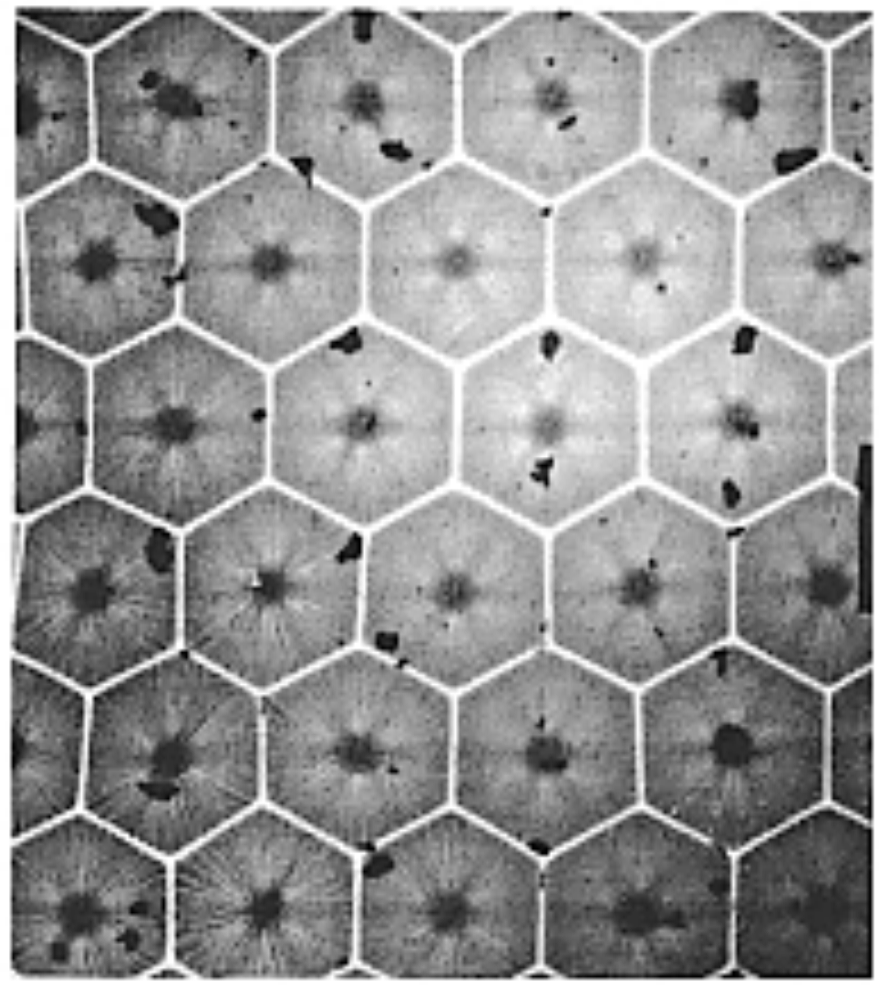
Nataf et al, 1981

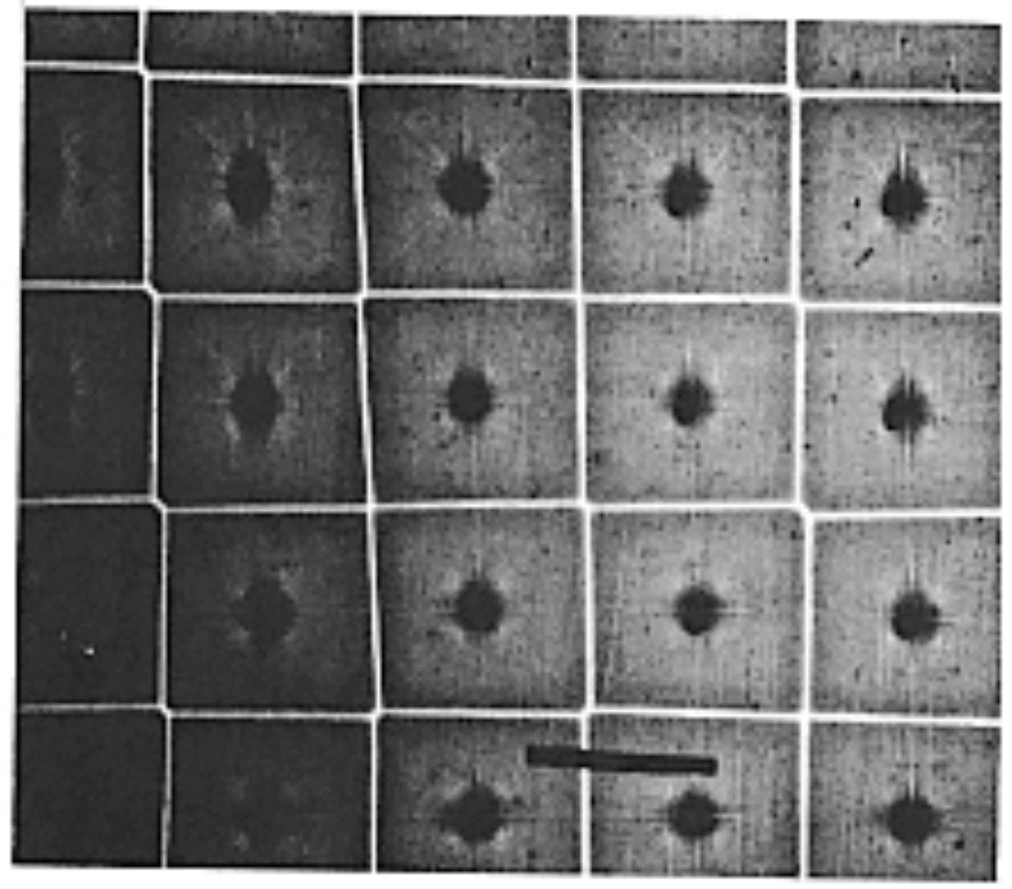
Convection patterns



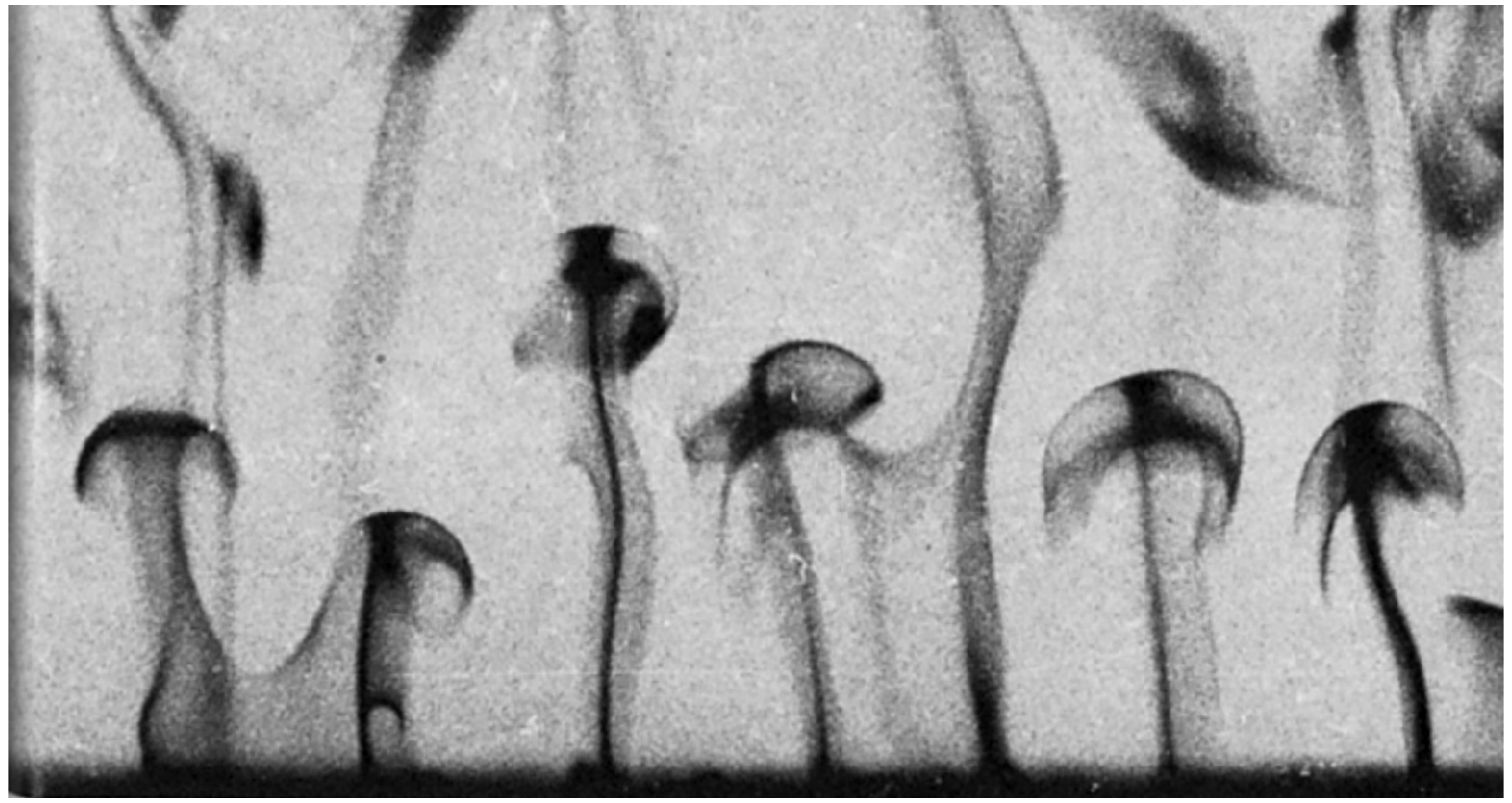




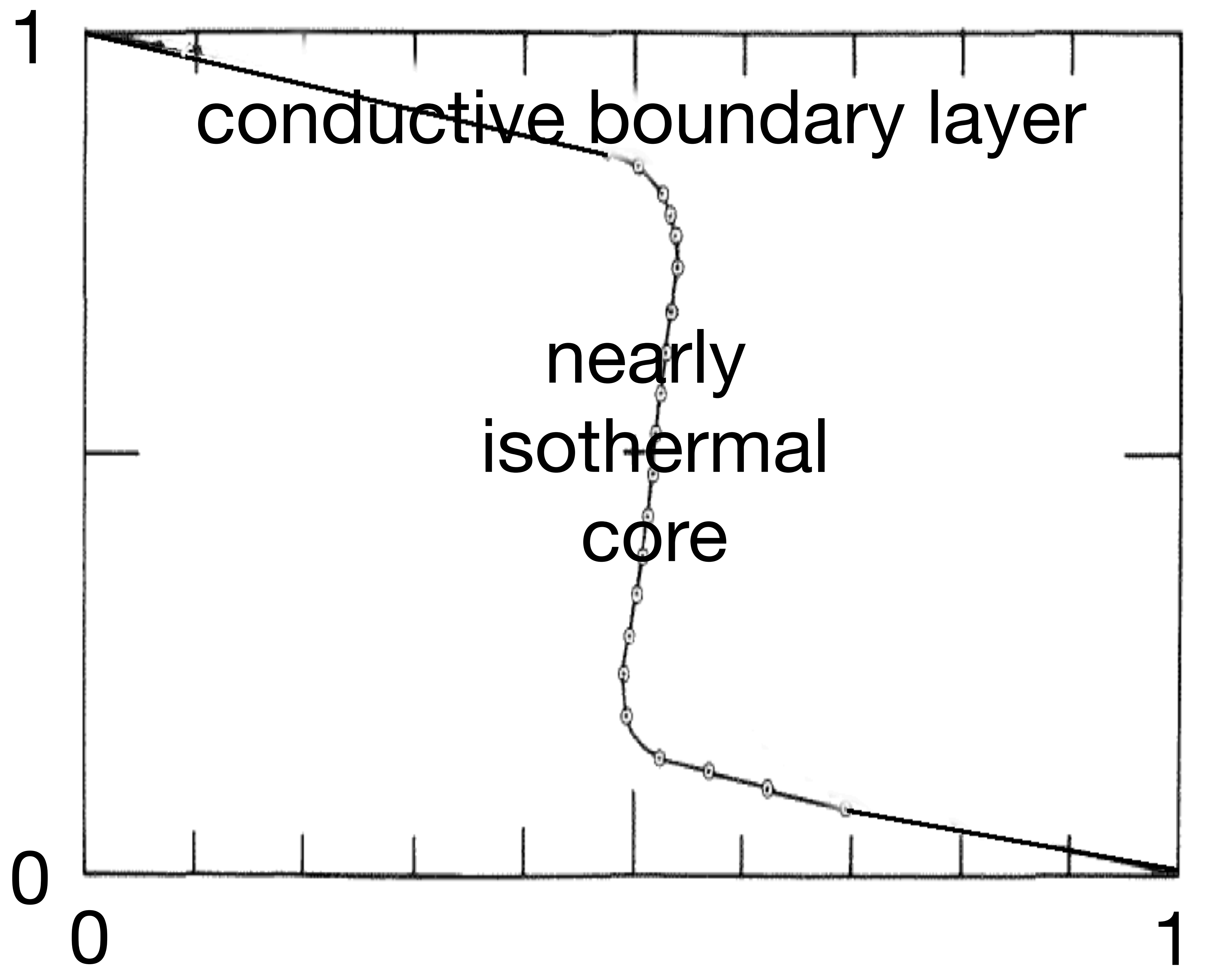




White, 1982



- Patterns are a great source of inspiration for artists, but often it's just as well to average them out and get the **horizontally averaged temperature profile** $\bar{T}(z)$. It usually looks like this:



1.4. Earth's interior

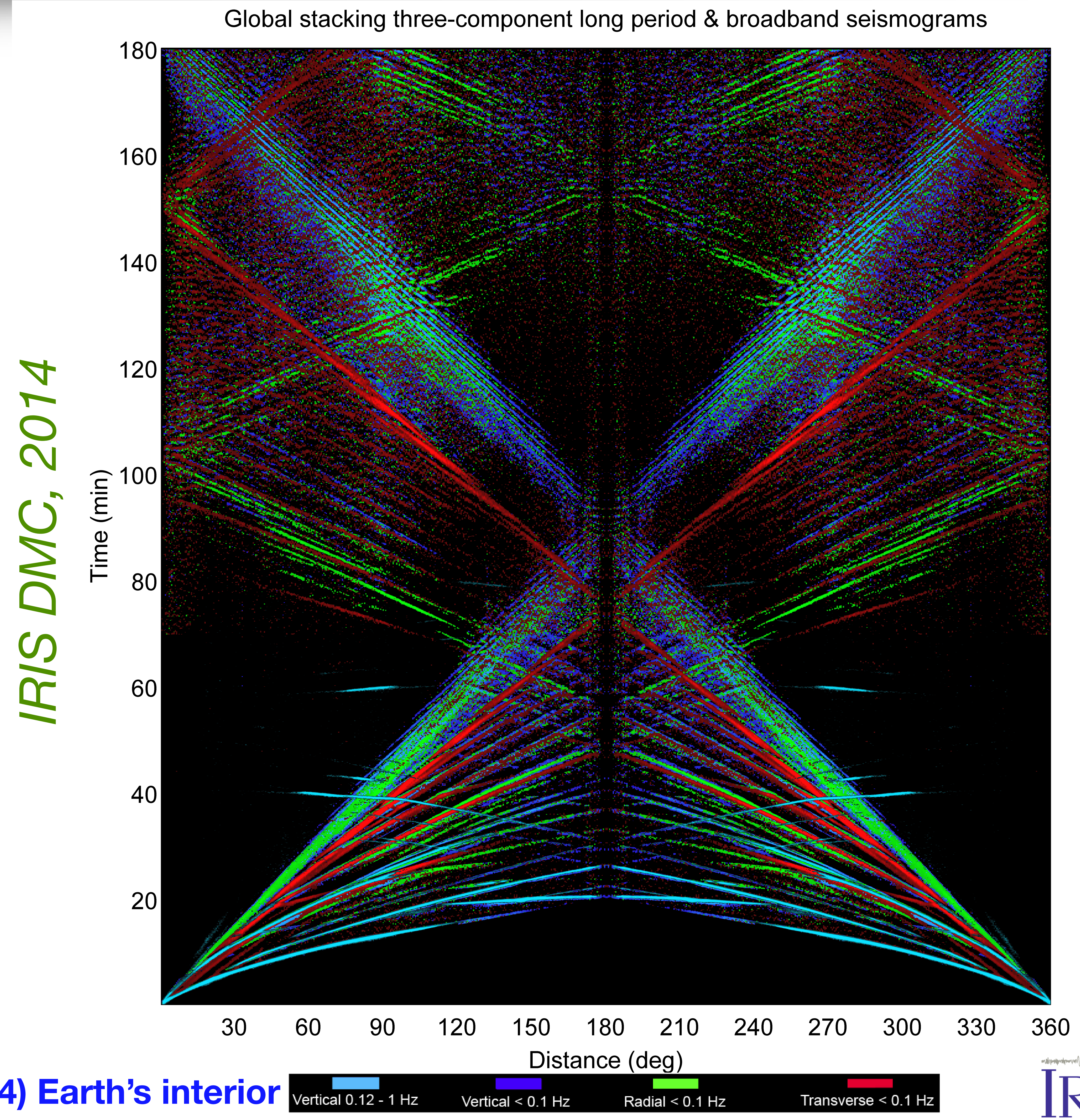
Earth's interior outline

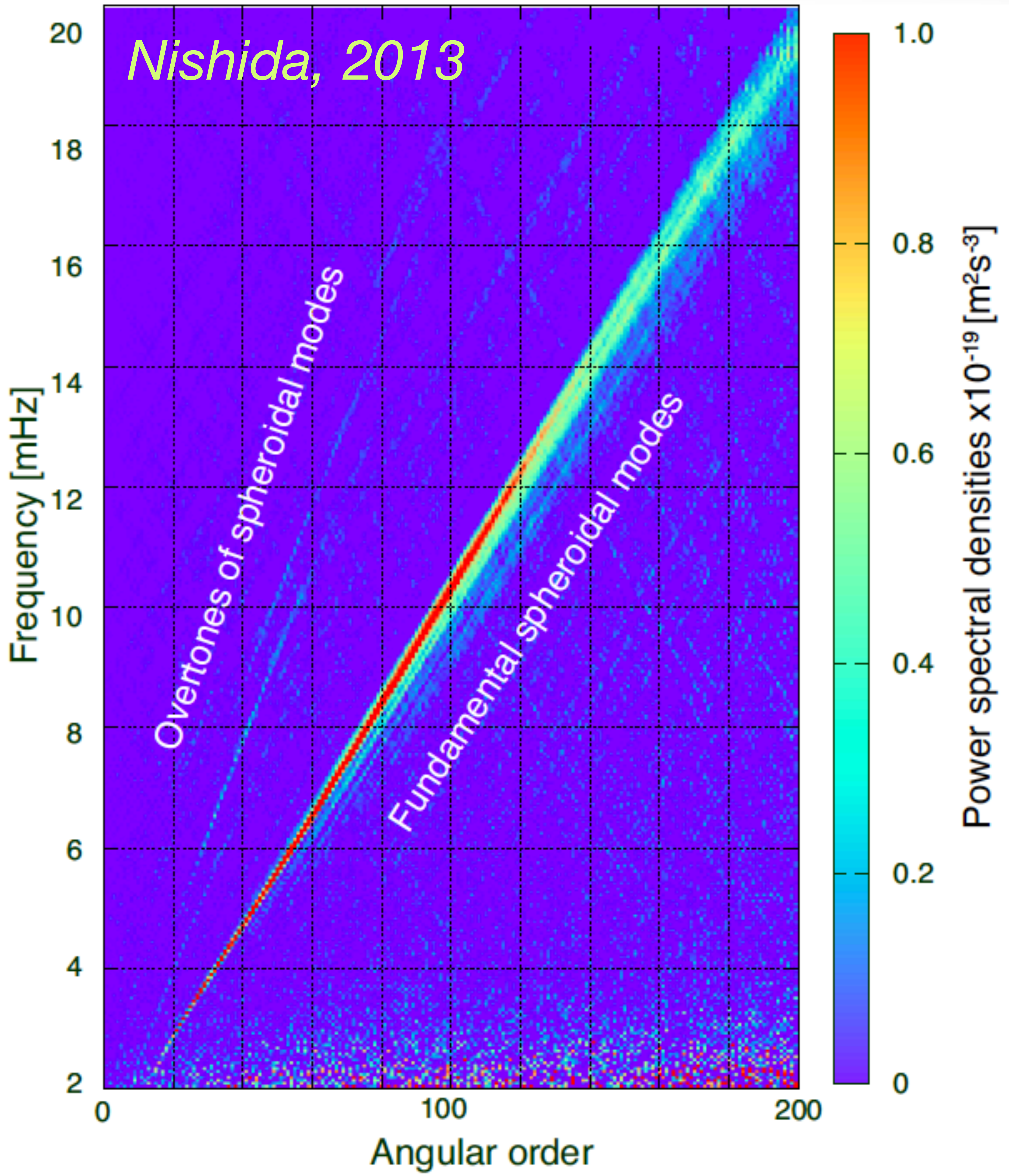
1.4. Earth's interior

- 1.4.1. Structure
- 1.4.2. Composition
- 1.4.3. Energy

1.4.1. Structure of the Earth’s interior

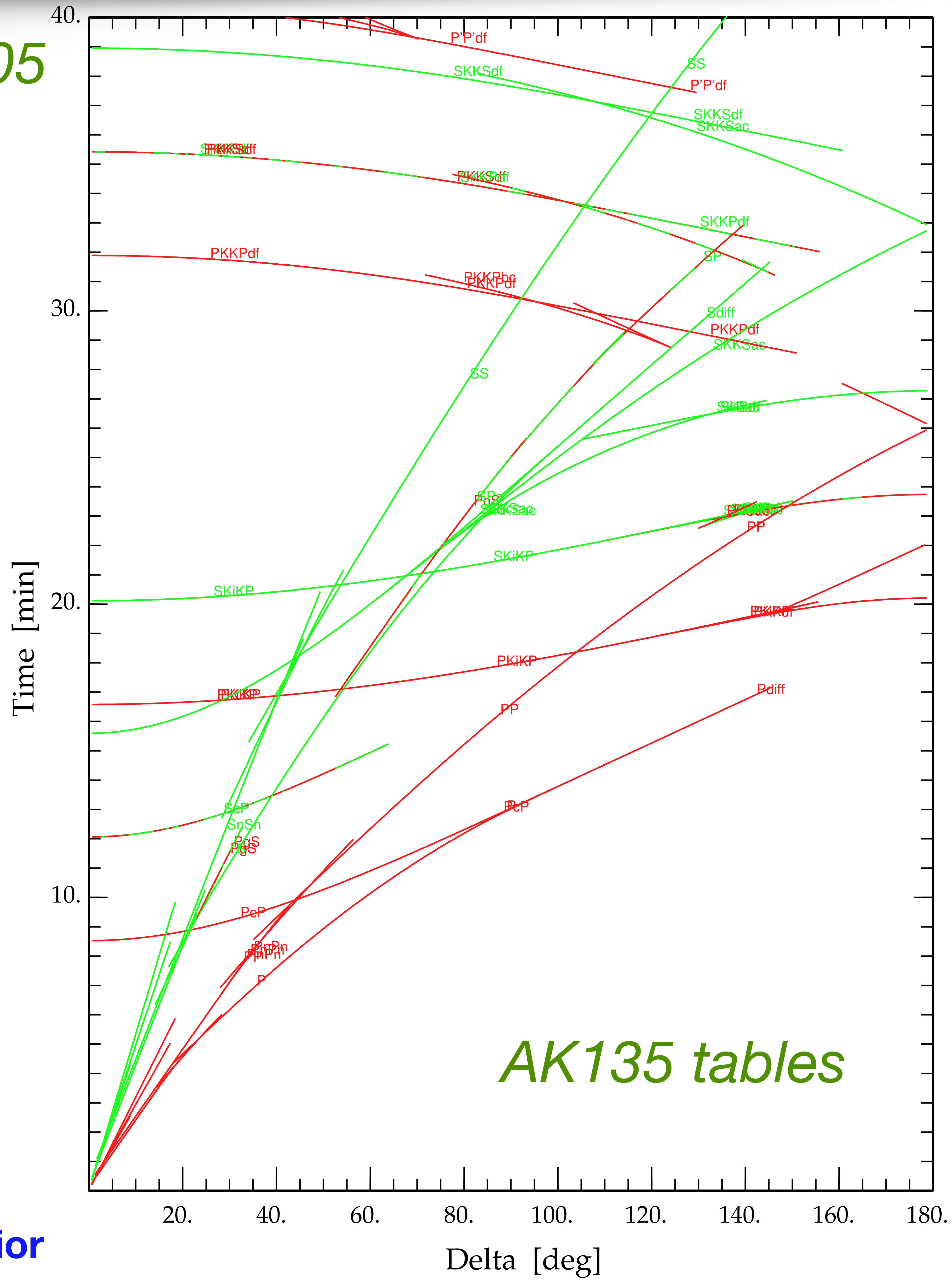
seismological radiography of the Earth

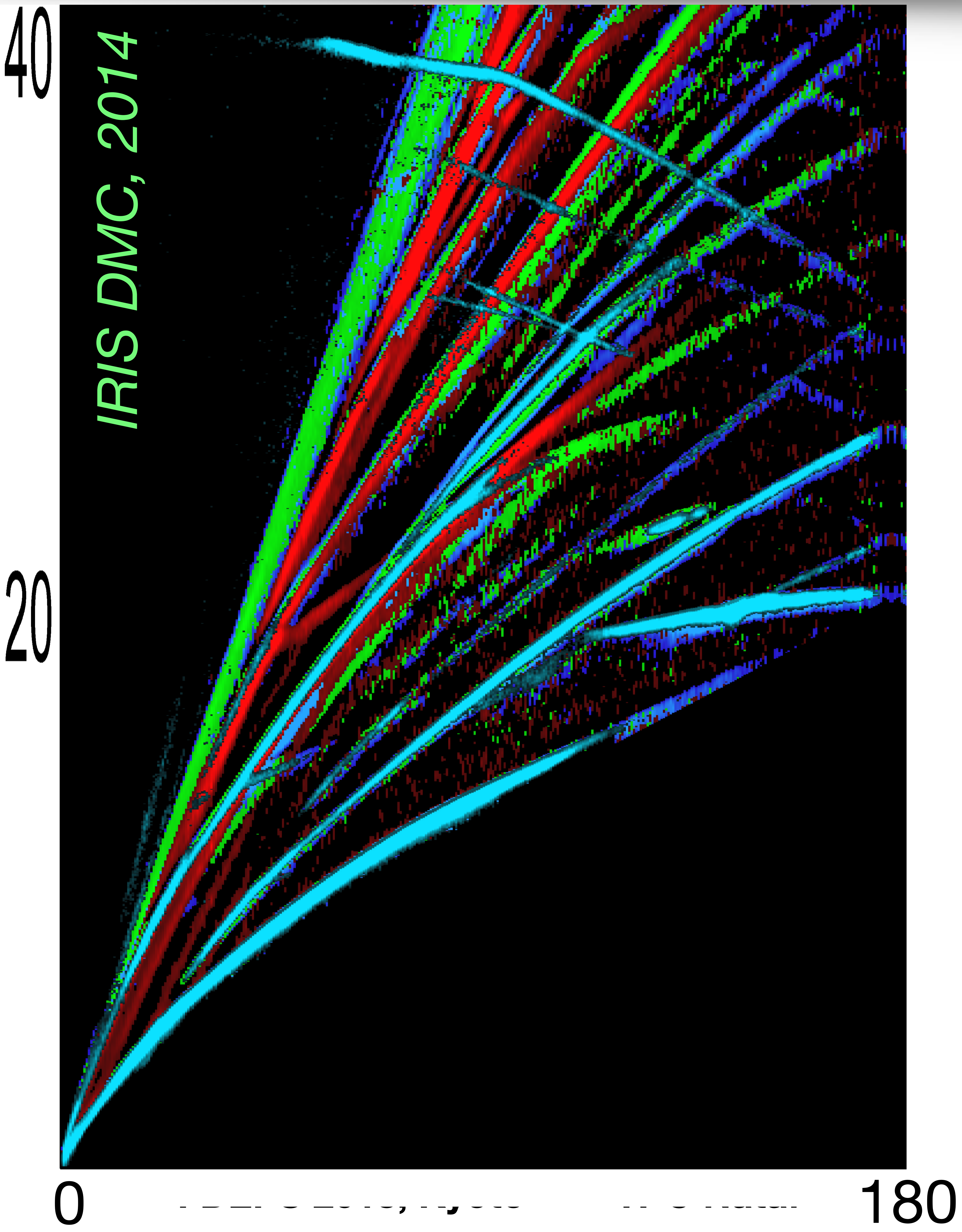




travel-times of seismic waves

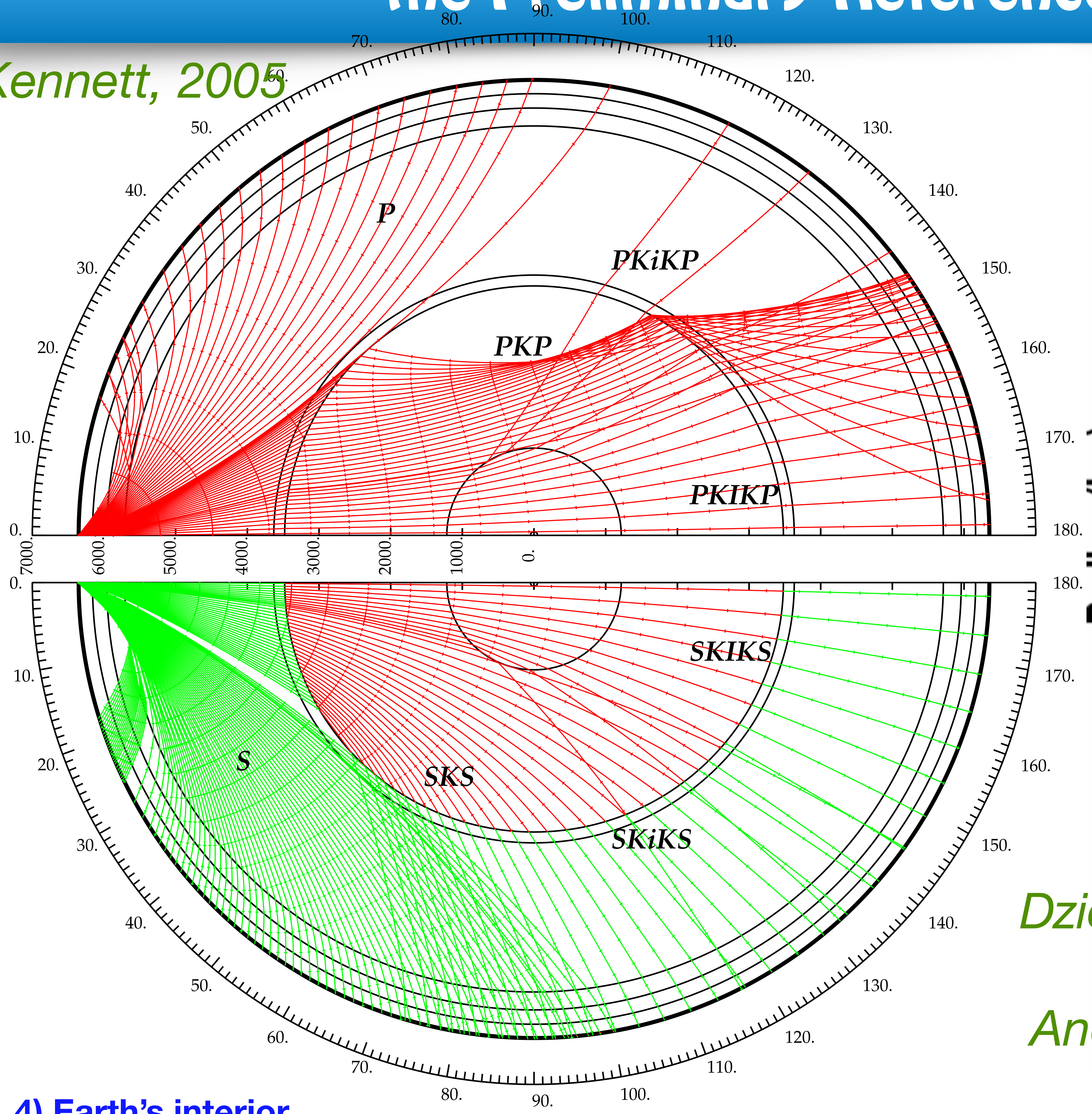
Kennett, 2005

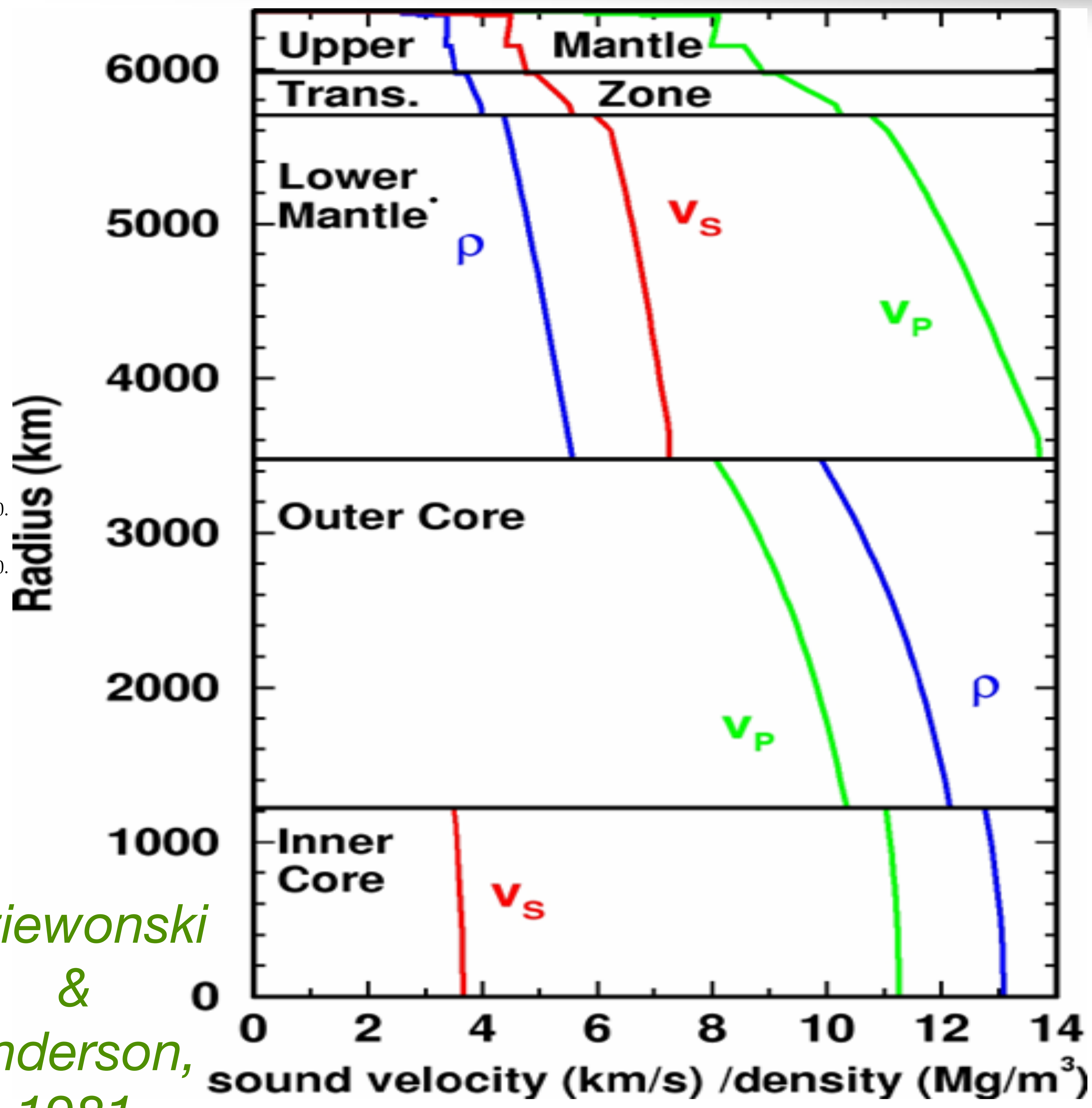




the Preliminary Reference Earth Model (PREM)

Kennett, 2005

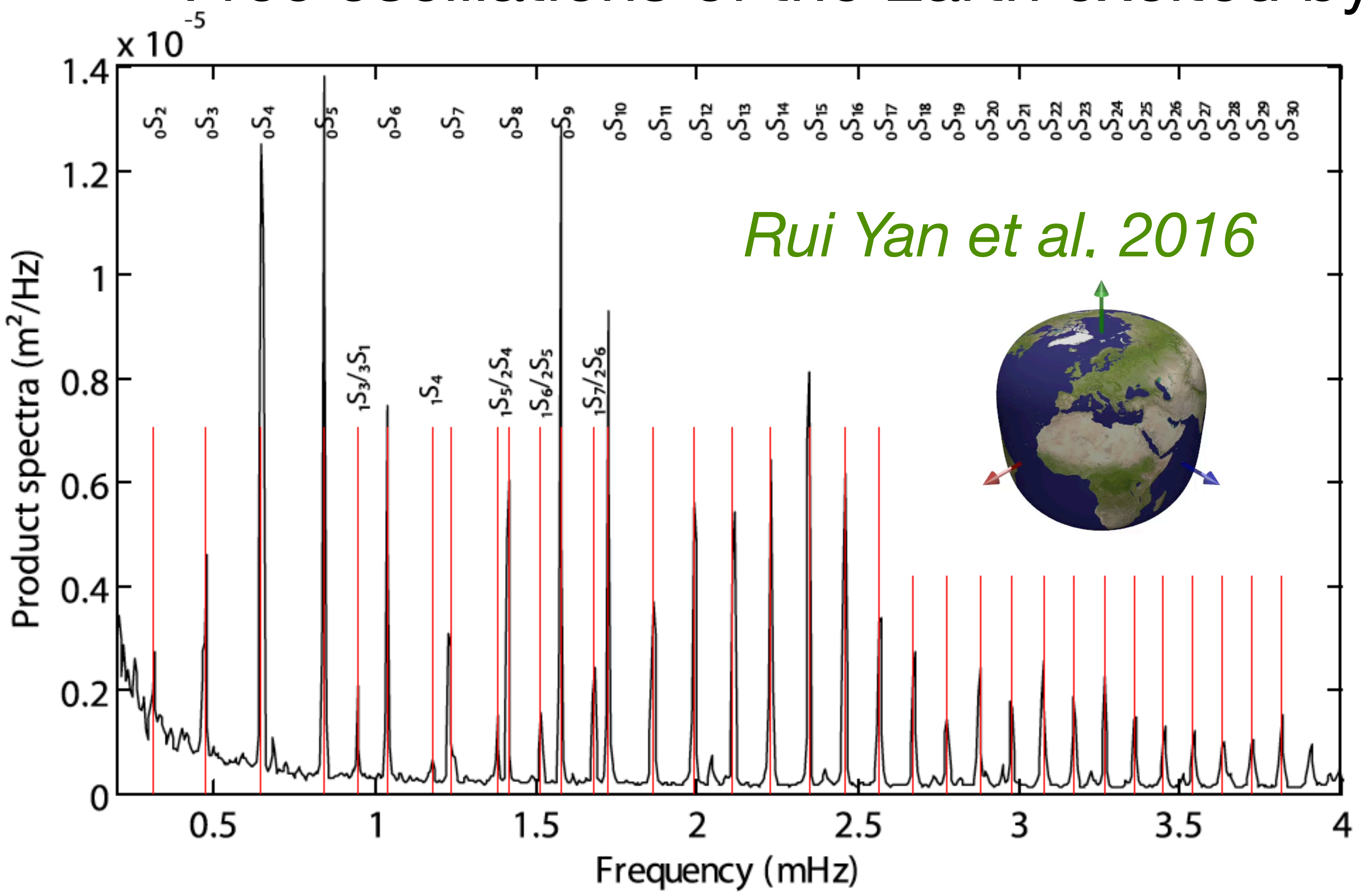


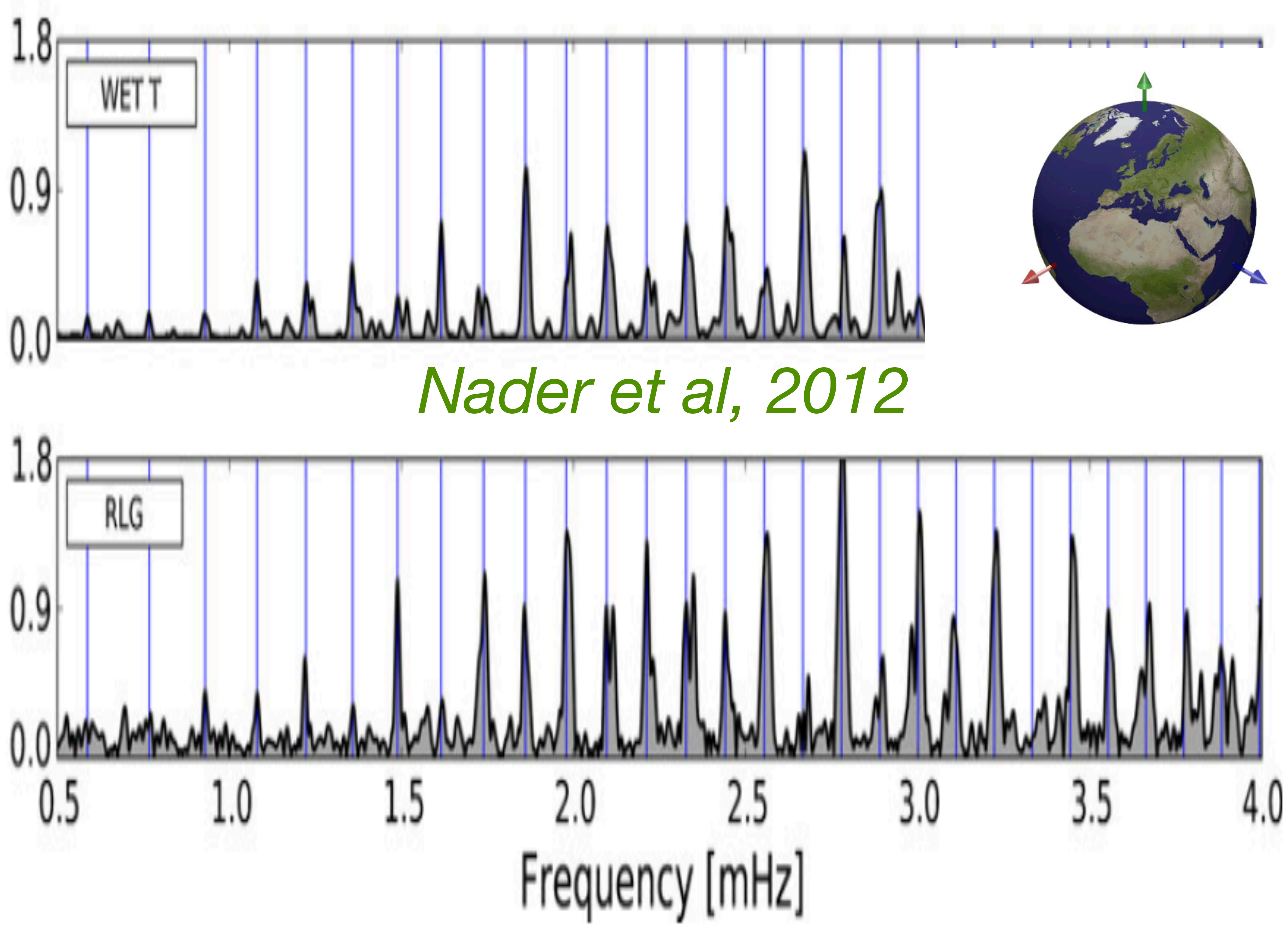


Dziewonski & Anderson, 1981

free oscillations (or normal modes) of the Earth

Free oscillations of the Earth excited by the 2011 Tohoku Mw 9.0 earthquake



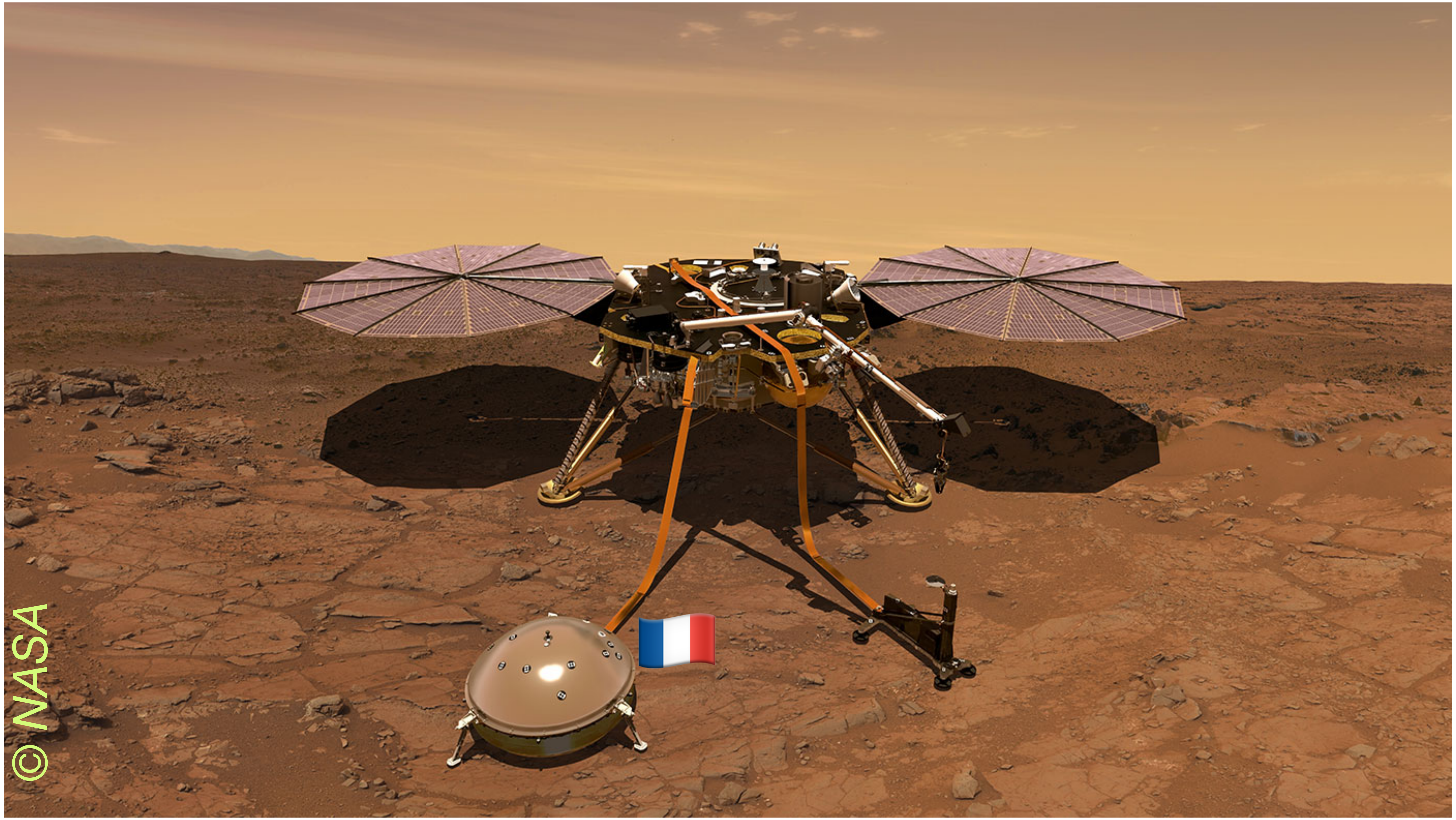


Spheroidal free oscillations observed in water levels from 43 wells in China.

Torsional free oscillations recorded by a ring laser rotation sensor in Europe.

NASA's InSight landed on Mars this morning !...

Will Mars be the next (and only?) other planet for which we determine the interior structure from seismology?



1.4.2. Composition of the Earth’s interior

A simplified view of the composition of the Earth's interior

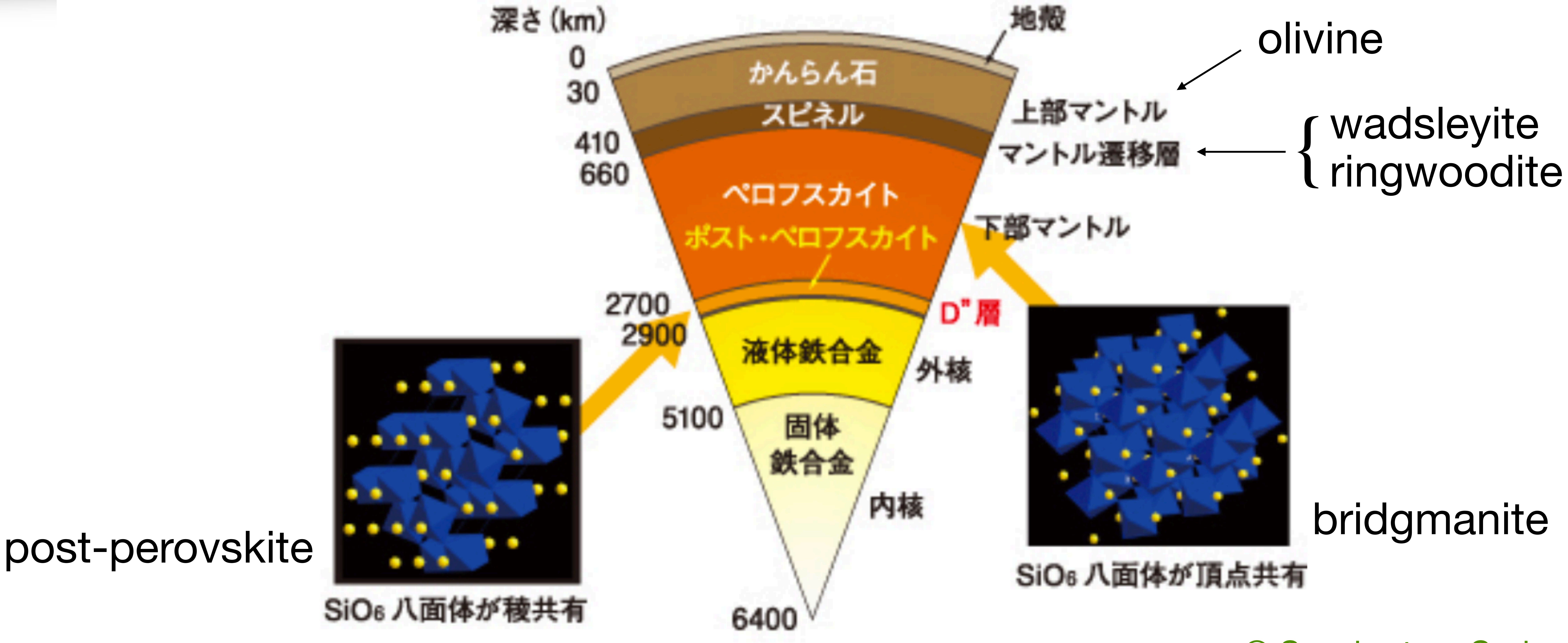
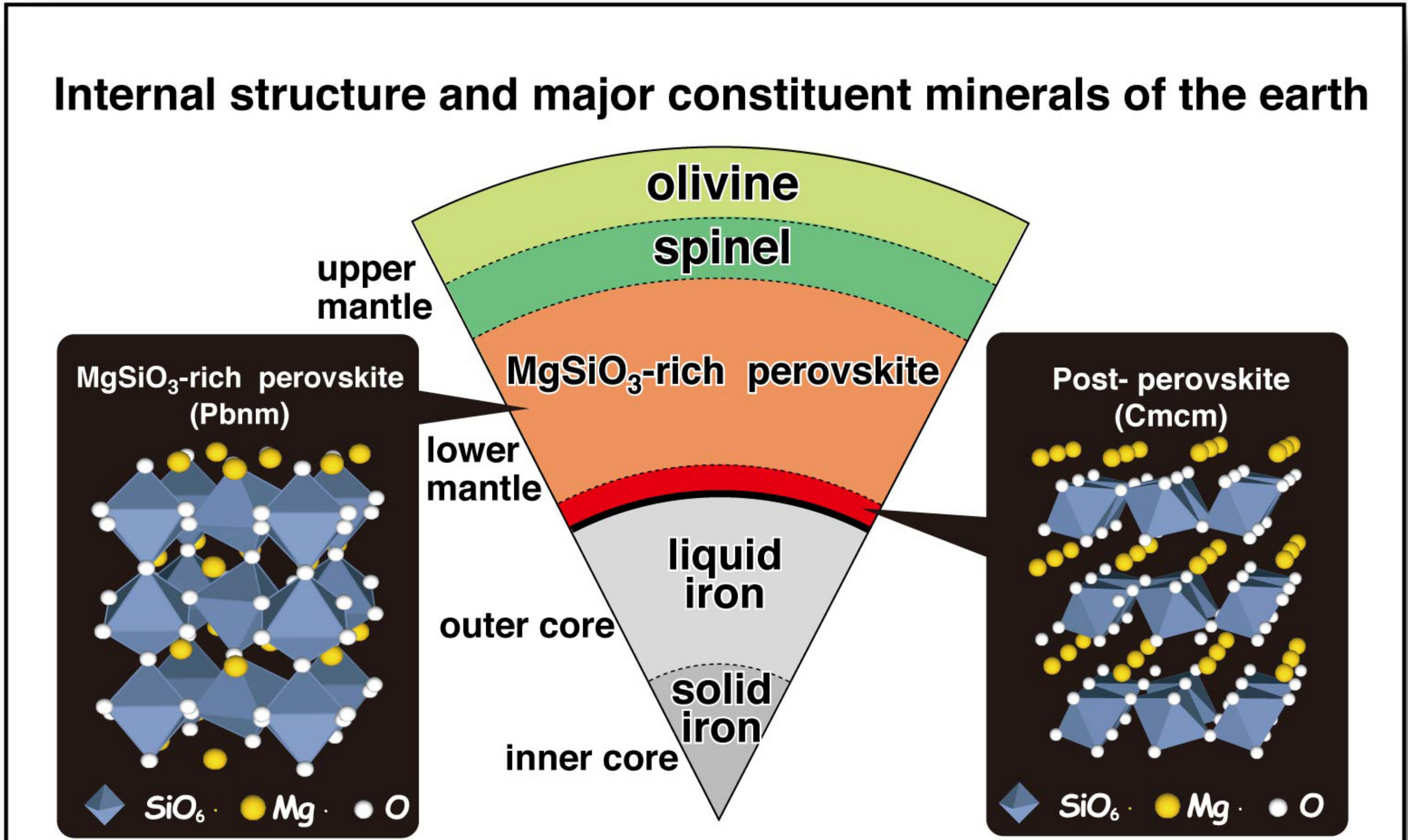


図1.地球内部の層構造

従来、下部マントルとD”層はペロフスカイトで構成されていると考えられていたが、今回の研究でD”層のポストペロフスカイトの存在が明らかになった。

A simplified view of the composition of the Earth's interior



Kei Hirose, 2013

1.4.3. Energy from the Earth

How much heat out of the Earth?

46 TW = 46 x 10^{12} W

Jaupart et al, 2015

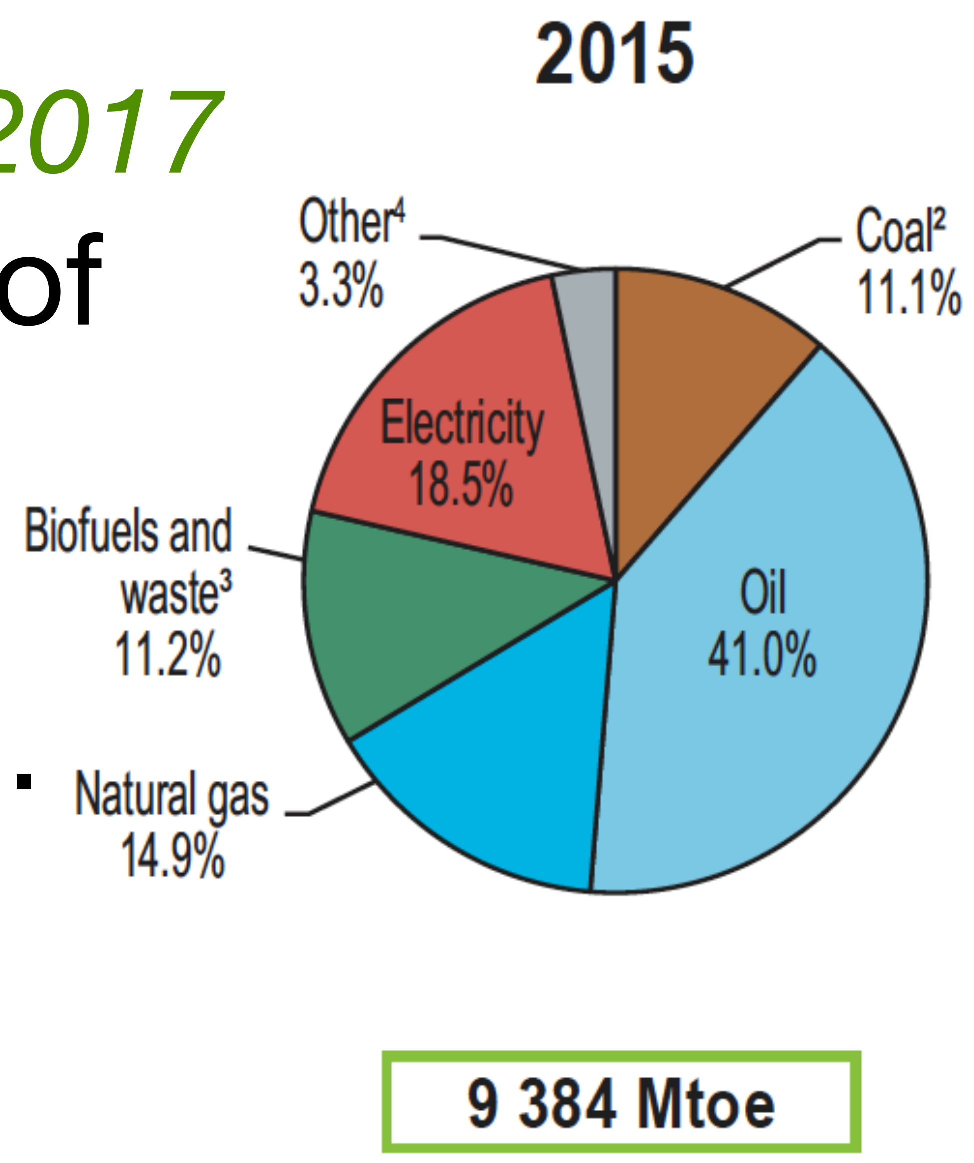
This looks like a large number, **but**:

IAE, 2017

- It's **only 3.7 times** the total power consumption of humanity as of 2015.

- It is **3800 less** power than what the Earth receives from the Sun.

- It corresponds to an internal production of only **42 W per cubic kilometer.**



Where is the heat coming from?

- Continental heat loss: 13-15 TW
- Convective mantle: 35-41 TW
- Core: 5-17 TW

Total: 43-49 TW

Jaupart et al, 2015

Uncertainties are still large!

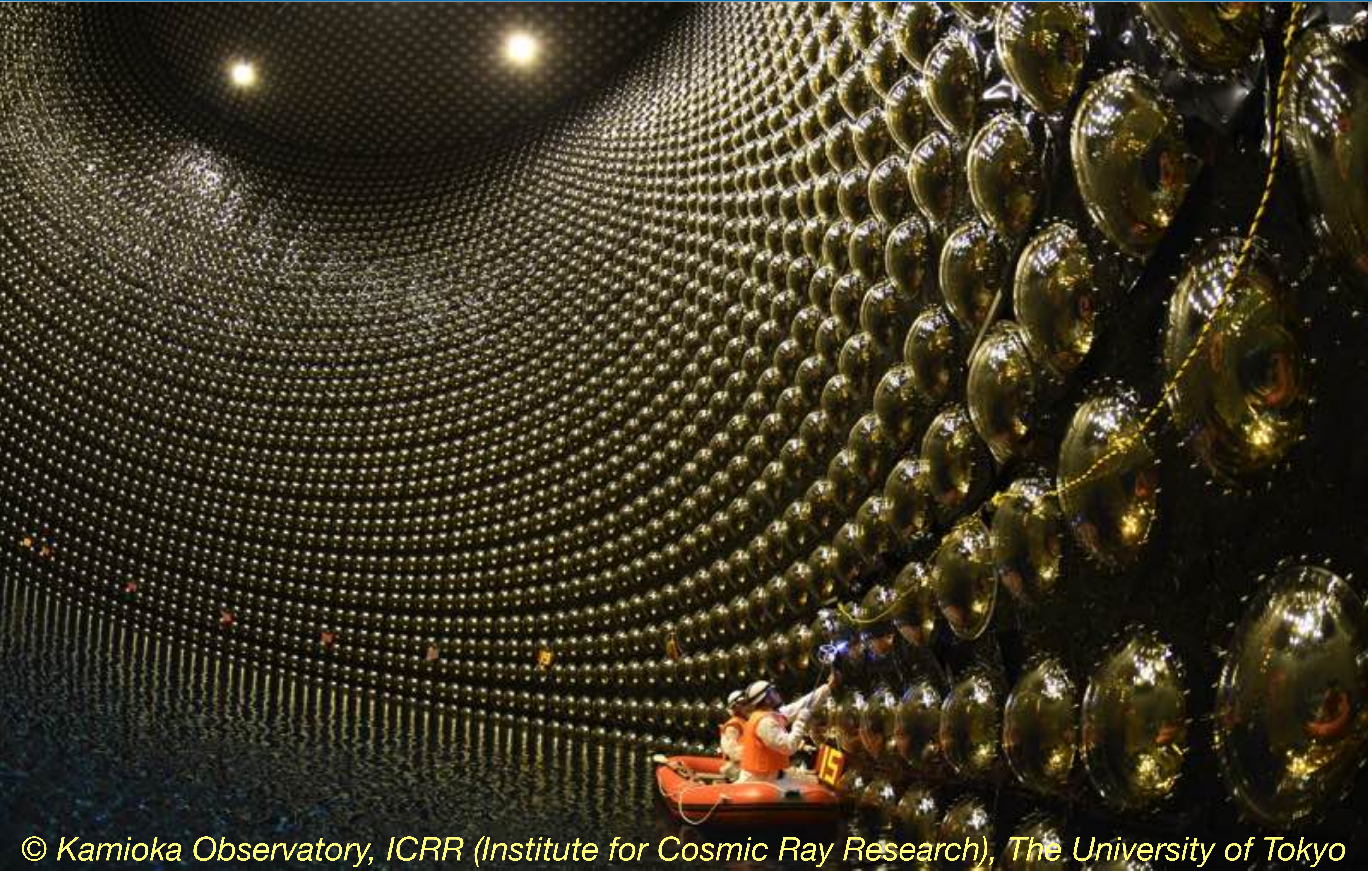
What is the origin of the heat?

- Secular cooling: 1-29 TW — cooling rate of 7-210 K/Ga
- Radioactive elements: 13-23 TW — ^{238}U, ^{235}U, ^{232}Th, ^{40}K
- Mechanical energy: < 0.1 TW?

Jaupart et al, 2015

Uncertainties are still large!

antineutrinos will help!



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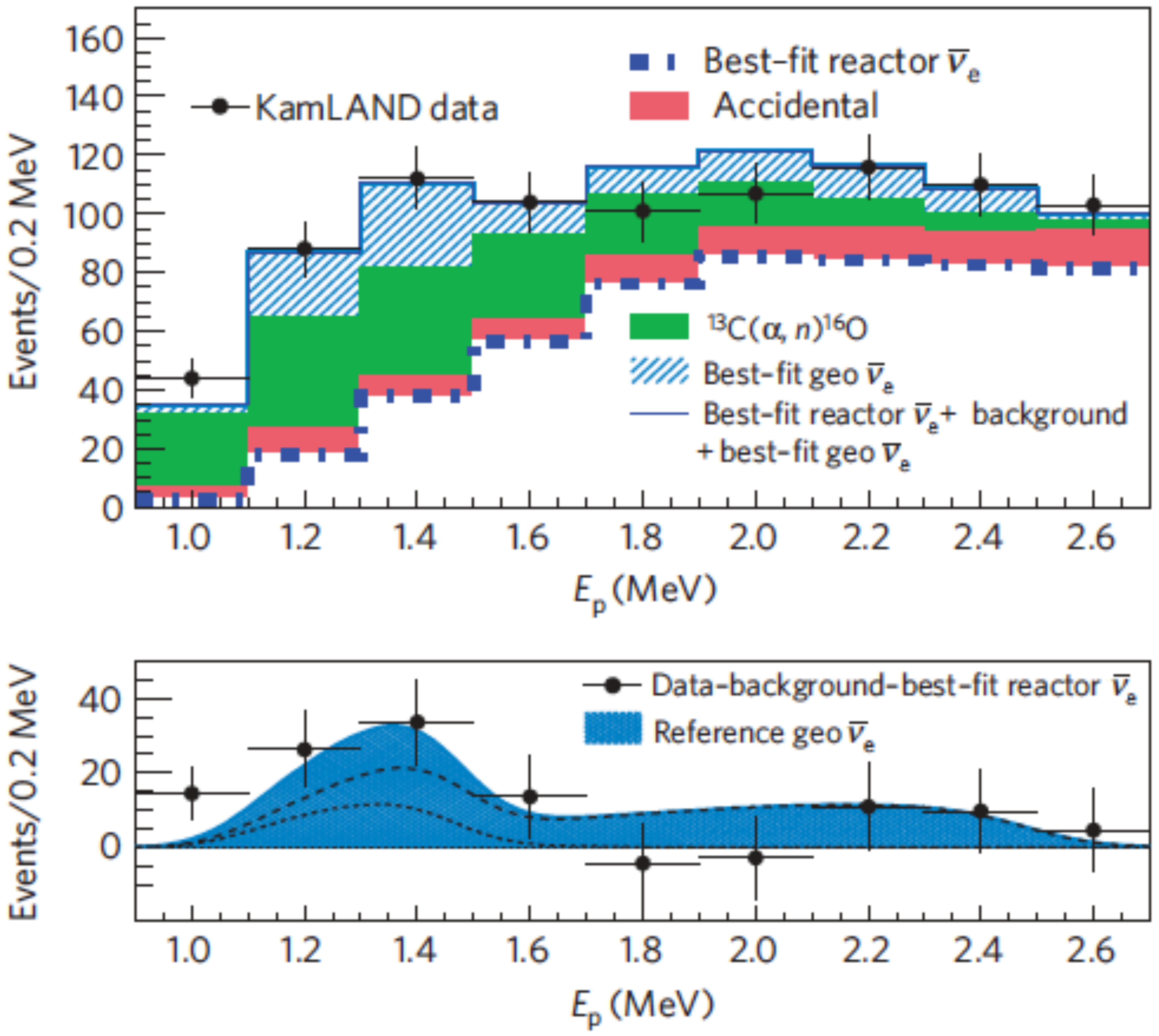
The disintegration chains of Uranium, Thorium and Potassium produce electron antineutrinos:

$$^{238}U \rightarrow \ldots \rightarrow \ldots + \bar{\nu}_e$$
$$^{232}Th \rightarrow \ldots \rightarrow \ldots + \bar{\nu}_e$$

These 'geoneutrinos' are detected in huge detectors such as the Japanese Super-Kamiokande by the inverse β-decay reaction:

$$\bar{\nu}_e + p \rightarrow e^+ + n$$

antineutrinos will help!



The KamLAND collaboration, 2011

It takes several years of events' collection to detect a few dizains of geoneutrinos!

So far, the results are consistent with geochemical models such as McDonough and Sun (1995), where present-day radioactive sources produce 19 TW of heat.

Great hope rests in on-going installations: Borexino, SNO+… and projects: Hyper-Kamiokande.

What is the origin of the heat?

- Secular cooling: 1-29 TW — cooling rate of 7-210 K/Ga
- Radioactive elements: 13-23 TW — ^{238}U, ^{235}U, ^{232}Th, ^{40}K
- Mechanical energy: < 0.1 TW?

Jaupart et al, 2015

- ***Neutrinos???***